

The University of Chicago

STUDIES OF THE INVERSE
PROBLEM OF THE CALCULUS OF
VARIATIONS

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1937

By
NATHAN ABRAHAM MOSCOVITCH

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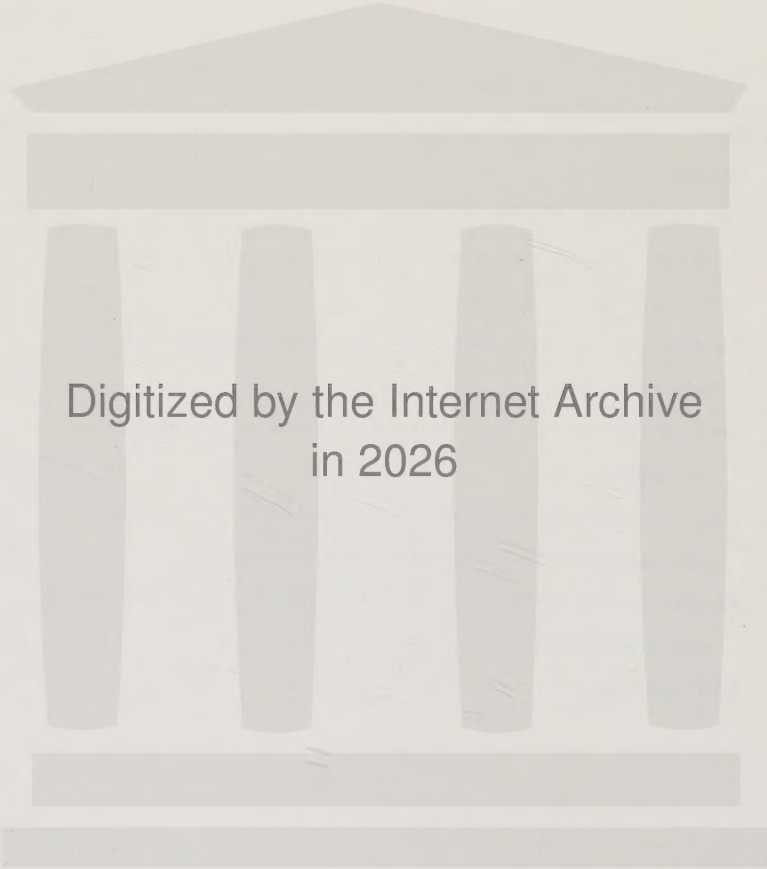
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STUDIES OF THE INVERSE PROBLEM OF THE CALCULUS OF VARIATIONS

1. Introduction. This paper deals mainly with two problems. The first is that of finding a problem of Lagrange in the calculus of variations which has a given function for its Hamiltonian function. The second is the determination of the relationship between a problem in the calculus of variations, its extremals and transversality, and an infinitesimal contact transformation with its path curves and transversality.

The former problem was first treated by C. Carathéodory [5, pp. 223-7].¹ However, there are two points in his treatment where improvements can be made. He assumes that in the Hessian matrix $\|H_{z_1 z_k}\|$ of the given Hamiltonian function $H(x, y, z)$, a particular m -rowed minor is always different from zero, whereas in the present paper only the weaker and more symmetric assumption is made that the matrix has rank $n - m$. Furthermore Carathéodory does not specify the regions in which his results hold. In this respect it was necessary to extend not only the final results but also some preliminary theorems [5, pp. 200-203] as is done in Section 4 below. The generalization of Carathéodory's method of finding a problem of Lagrange having a given Hamiltonian function is presented in Section 3. The result is not unique so that Section 4 is devoted to a discussion of the relation between different problems of Lagrange having the same Hamiltonian func-

¹The numbers in square brackets refer to the bibliography presented at the end of the paper.

tion. It is proved that such problems are equivalent according to a suitable definition. This is an extension of the work of Carathéodory who proved what is approximately the converse of this theorem [10, pp. 353].

In the fifth section there is presented an application of the preceding results to the inverse problem in the calculus of variations and also a characterization of the family of extremals of a problem of the calculus of variations.

The second part of the paper unifies and extends some results attained by Douglas [7, pp. 401-409] and Rawles [8, p. 765]. Necessary and sufficient conditions that a transversality belong to a calculus of variations problem, and to an infinitesimal contact transformation, are developed. Conditions are also developed which are necessary and sufficient in order that a calculus of variations problem, and an infinitesimal contact transformation should have the same transversality and that moreover the extremals of the calculus of variations problem should coincide with the path curves of the infinitesimal contact transformation. These are the results of Section 6. Section 7 discusses the problem of characterizing the composite of extremals and transversality of a calculus of variations problem. The results of these two sections include those of Douglas and Rawles, but by the methods presented here, their conclusions are obtained more directly.

2. Two auxiliary theorems. In order to discuss the generalization of the theory of Carathéodory mentioned above in the introduction to this paper, it will be necessary to prove two auxiliary theorems. The first is an extension of a theorem originally proved by Bliss [4, pp. 307, 312] and may be stated as follows:

THEOREM 2:1. If a matrix $\|a_{\sigma j}(x)\|$ of functions $a_{\sigma j}(x)$ ($\sigma = 1, \dots, m < n$; $j = 1, \dots, n$) continuous on an interval $x_1 x_2$, has rank $r \leq m$ at each point of the interval, then $m - r$ additional columns of polynomials $b_{\sigma \tau}(x)$ ($\sigma = 1, \dots, m$; $\tau = 1, \dots, m - r$) can be selected so that the matrix $\|a_{\sigma j}(x), b_{\sigma \tau}(x)\|$ has rank m at each point of the interval $x_1 x_2$.

To prove this, consider the linear equations

$$(2:1) \quad u_{\sigma} a_{\sigma j}(x) = 0$$

in which the repeated subscript indicates summation as in tensor analysis. Since one of the determinants of order r of the matrix $\|a_{\sigma j}(x)\|$ is different from zero at x_1 , we may select $m - r$ of the functions $u_{\sigma}(x)$ arbitrarily except that they are continuous functions of x in a neighborhood of x_1 , not vanishing simultaneously, and then determine the remaining r of the functions u_{σ} in this neighborhood as solutions of the equations (2:1). Let ξ be the least upper bound of values x_3 defining intervals $x_1 x_3$ on which the equations (2:1) have continuous solutions not vanishing simultaneously. There exists a δ -neighborhood of ξ in which one r -rowed minor of the matrix $\|a_{\sigma j}(x)\|$ remains different from zero. Select x_3 in this neighborhood so that $x_1 < x_3 < \xi$. The equations (2:1) have continuous solutions $u_{\sigma}(x)$ on $x_1 x_3$ and the definition of any $m - r$ of them can be extended so that these $m - r$ extended functions are continuous and do not vanish simultaneously on the interval from x_1 to $\xi + \delta$. If the $m - r$ extended functions are properly chosen, then the equations (2:1) define the remaining r functions $u_{\sigma}(x)$ uniquely on the interval from x_3 to $\xi + \delta$, and these latter do not vanish simultaneously. It follows then that ξ cannot be the least upper bound described above unless ξ is at x_2 , and if ξ is at x_2 , there exist solu-

tions $u_\sigma(x)$ of equations (2:1), continuous and not vanishing simultaneously on the whole interval x_1x_2 .

The matrix $\|a_{\sigma j}(x), u_\sigma(x)\|$ has rank $r + 1$ everywhere on the interval x_1x_2 . For, if not, suppose at x_3 , the rank is r , where $x_1 \leq x_3 \leq x_2$. Then the equations

$$a_{\sigma j}(x_3)v_j + u_\sigma(x_3)v = 0$$

have $n - r + 1$ linearly independent sets of solutions $v_{j\rho}$, v_ρ ($\rho = 1, \dots, n - r + 1$). Hence, from the last equations,

$$u_\sigma(x_3) a_{\sigma j}(x_3) v_{j\rho} + u_\sigma(x_3) u_\sigma(x_3) v = 0$$

and equations (2:1) together with the fact that the values $u_\sigma(x_3)$ are not all zero imply that $v_\rho = 0$. Consequently the sets $v_{j\rho}$ form $n - r + 1$ linearly independent solutions of the equations

$$a_{\sigma j}(x_3) v_j = 0,$$

which implies that the matrix $\|a_{\sigma j}(x)\|$ has rank $r - 1$ at x_3 , contrary to hypothesis.

The above process may now be repeated $m - r$ times and $m - r$ columns of continuous functions $b_{\sigma\tau}(x)$ are thus obtained, for which the matrix $\|a_{\sigma j}(x), b_{\sigma\tau}(x)\|$ has rank m . The statement of the theorem is justified since a continuous function may be approximated as closely as desired by a polynomial. In the sequel we shall use the theorem for the special case when $m = n$, $r = n - m$.

We may now prove the following second auxiliary theorem:

THEOREM 2:2. If a symmetric matrix $\|a_{ij}\|$ ($i, j = 1, \dots, n$) has rank $n - m$, and if the matrix $\|a_{ij}, b_{\rho i}\|$ ($i, j = 1, \dots, n$; $\rho = 1, \dots, m$) in which the subscript i varies in the

columns, and j and β in the rows, has rank n , then the square matrix

$$\begin{vmatrix} a_{1j} & b_{\beta j} \\ b_{\alpha j} & 0 \end{vmatrix},$$

($\alpha = 1, \dots, m$), in which i and α vary in the columns, and j and β in the rows, is non-singular.

Since the matrix $\|a_{1j}\|$ has rank $n - m$ there exist m linearly independent sets of constants $d_{\alpha 1}$ such that

$$a_{1j}d_{\alpha j} = 0, \quad d_{\alpha 1}a_{1j} = 0.$$

Since the matrix $\|a_{1j} \ b_{\beta 1}\|$ has rank n , it follows that, if there exists a set of constants c_1 such that $c_1 a_{1j} = 0$, $c_1 b_{\alpha 1} = 0$, then $c_1 = 0$. Now suppose the theorem is false. Then there exist constants $e_1, e_{n+\beta}$, not all zero, such that

$$(2:2) \quad a_{1j}e_j + b_{\beta 1}e_{n+\beta} = 0, \quad b_{\alpha j}e_j = 0.$$

But the constants $e_j, e_{n+\beta}$ must be a linear combination of every system $u_{j\beta}, v_{\alpha\beta}$ of m linearly independent solutions of the equations

$$a_{1j}u_j + b_{\alpha 1}v_{\alpha} = 0.$$

One such system is $(u_{j\beta}, v_{\alpha\beta}) = (d_{\beta j}, 0)$, and it is evident that $e_{n+\beta} = 0$. But from equations (2:2) and the result proved above concerning the constants c_1 it follows that $e_j = 0$ also. We have now reached a contradiction, and the theorem is therefore proved.

3. Problems of the calculus of variations with a given Hamiltonian function. Consider a region R of the space of $2n + 1$ dimensional points (x, y, z) where y and z stand for n dimensional sets $(y_1, \dots, y_n)$ and $(z_1, \dots, z_n)$ respectively, and let

$H(x, y, z)$ be a function with the following properties:

(a) H is of class $C^{(4)}$ in the region R ,

(b) the matrix $\|H_{z_1 z_k}\|$ ($i, k = 1, \dots, n$) of second derivatives of H with respect to the variables z_1 , is of rank $n - m$ at every point of R . By a function of class $C^{(n)}$ we mean one which has all of its partial derivatives continuous, up to and including those of order n .

Consider also an arc E

$$y_1 = y_1(x), \quad z_1 = z_1(x), \quad (x_1 \leq x \leq x_2)$$

with functions $y_1(x)$, $z_1(x)$ continuous and having continuous first derivatives in a neighborhood of the interval $x_1 x_2$, having all of its points interior to the region R , and satisfying the differential equations

$$(3.1) \quad dy_1/dx = H_{z_1}.$$

Let us denote by E_1 , the projection of the arc E on the space of $n + 1$ dimensional points (x, y) . We wish to determine a problem of Lagrange in the calculus of variations defined in a neighborhood of E_1 and having the function $H(x, y, z)$ as its Hamiltonian function. This is the problem described above in the introduction to this paper.

Along the arc E the matrix $\|H_{z_1 z_k}\|$ is a matrix of continuous functions of x . By Theorem (2.1), there exist therefore polynomials $w_{1\beta}(x)$ ($\beta = 1, \dots, m$) such that the matrix $\|H_{z_1 z_k} w_{1\beta}\|$ has rank n along E . Let us define m functions linear in the variables z_1 by means of the equations

$$\psi_{\beta}(x, y, z) = z_1 w_{1\beta}(x).$$

Then the preceding matrix may be written in the form

$\|H_{z_1 z_k} \psi_{\beta z_1}\|$. Let us now consider the equations

$$(3:2) \quad y_1' = H_{z_1}(x, y, z) + \nu_{\alpha} \psi_{\alpha z_1}(x, y, z), \quad \lambda_{\alpha} = \psi_{\alpha}(x, y, z).$$

These equations have as solutions the values $(x, y, y', \lambda, z, \nu)$ for which x, y_1, y_1', z_1 , with y_1' the derivatives of the functions $y_1(x)$ belonging to the arc E , and for which $\nu_{\alpha} = 0$ and λ_{α} is determined by the second equations (3:2). We call this set of values the set belonging to E . The functional determinant of the second members with respect to the variables z_1, ν_{α} is

$$\begin{vmatrix} H_{z_1 z_k} & \psi_{\beta z_1} \\ \psi_{\alpha z_k} & 0 \end{vmatrix}$$

which by Theorem (2:2) is different from zero along the arc E . Hence, in accordance with well known theorems concerning implicit functions, the equations (3:2) have solutions z_1, ν_{α} , defined and of class C''' in a neighborhood N of the values (x, y, y', λ) belonging to E , and the solutions may be written in the form

$$(3:3) \quad z_1 = Z_1(x, y, y', \lambda), \quad \nu_{\alpha} = \phi_{\alpha}(x, y, y', \lambda).$$

Let us substitute the functions (3:3) in the equations (3:2) and differentiate the resulting identities with respect to the variables y_j', λ_{β} . Since the functions ψ_{β} are linear in the variables z_1 , we obtain the following relations:

$$(3:4) \quad \begin{aligned} H_{z_1 z_k} Z_{ky_j'} + \psi_{\beta z_1} \phi_{\beta y_j'} &= \delta_{1j}, \\ H_{z_1 z_k} Z_{k\lambda_{\alpha}} + \psi_{\beta z_1} \phi_{\beta \lambda_{\alpha}} &= 0, \\ \psi_{\beta z_k} Z_{ky_j'} &= 0, \quad \psi_{\beta z_k} Z_{k\lambda_{\alpha}} = \delta_{\beta\alpha}. \end{aligned}$$

Let us now consider the equations

$$(3:5) \quad H_{z_1 z_k}(x, y, Z)u_k + \psi_{\beta z_1}(x, y, Z)v_{\beta} = 0$$

in which the arguments (x, y, y', λ) of the derivatives of H and ψ_{β} belong to a particular point (x, y, y', λ) of a neighborhood N_1 chosen so small that in it the rank of the matrix $\|H_{z_1 z_k} \psi_{\beta z_1}\|$ remains equal to m . By well known theorems on linear equations these equations have a set of m linearly independent solutions of the form (u_k, v_{β}) , and every solution can be expressed as a linear combination of each such set. Similarly, the equations

$$H_{z_1 z_k}(x, y, Z)u_k = 0$$

have m linearly independent sets of solutions $u_{k\alpha}$ ($\alpha = 1, \dots, m$). Then the system $u_{k\alpha}, v_{\beta\alpha} = 0$ forms a system of m linearly independent solutions of equations (3:5). From the second of equations (3:4), we see that the sets $z_{k\lambda\alpha}, \phi_{\beta\lambda\alpha}$ satisfy equations (3:5), and hence they must be a linear combination of the sets $u_{k\alpha}, v_{\beta\alpha} = 0$. Therefore the derivatives $\phi_{\beta\lambda\alpha}$ vanish at every point (x, y, y', λ) in N_1 and we find the following result:

LEMMA 3:1. When the functions $H(x, y, z)$ and $\psi_{\alpha}(x, y, z)$ have the properties described in the preceding paragraphs, the equations (3:2) have solutions

$$z_1 = Z_1(x, y, y', \lambda), \quad v_{\alpha} = \phi_{\alpha}(x, y, y')$$

which are defined and of class $C^{(3)}$ in a neighborhood N_1 of the values (x, y, y', λ) belonging to the arc E in equations (3:2). It should be noted that the functions ϕ_{α} are independent of the variables λ_{α} .

Now let us define a function $F(x, y, y', \lambda)$ by means of the equation

$$(3:6) \quad F(x, y, y', \lambda) = -H[x, y, Z(x, y, y', \lambda)] + y_1' Z_1(x, y, y', \lambda).$$

We notice that $F_{\lambda_\alpha} = -H_{z_1} Z_{1\lambda_\alpha} + y_1' Z_{1\lambda_\alpha} = \phi_\alpha(x, y, y')$, if we make use of equations (3:2), (3:3) and the last of (3:4). This shows that the function F is linear in the variables λ_α , so that it may be expressed in the form

$$(3:7) \quad F(x, y, y', \lambda) = f(x, y, y') + \lambda_\alpha \phi_\alpha(x, y, y').$$

We note also from equation (3:6) that

$$(3:8) \quad F_{y_1'} = -H_{z_j} Z_{jy_1'} + y_j' Z_{jy_1'} + z_1 = z_1,$$

by making use of equations (3:2) and the first of (3:4).

Now let us consider a problem of Lagrange in the calculus of variations for which it is required to minimize the integral

$$I = \int_{x_1}^{x_2} f(x, y, y') dx$$

in the class of admissible arcs satisfying the differential equations $\phi_\alpha(x, y, y') = 0$. In order that the original arc E define in xy_1 space an arc of this class, it is necessary and sufficient that E should satisfy the differential equation (3:1). This is an obvious consequence of equations (3:3) and (3:2).

Let us assume that E satisfies this condition and let us denote by E_1 the projection $y_1 = y_1(x)$ of the arc E . If we set up, in a neighborhood of the arc E_1 , the Hamiltonian function $H_1(x, y, z)$ for the problem under consideration, then we propose to show that $H_1 = H$. The value of $F(x, y, y', \lambda)$ for this problem is that given in equation (3:6) or (3:7). For this function we have $F_{y_1'} = z_1$ as indicated in equation (3:8). The equations $z_1 = F_{y_1'} = z_1$, $0 = \phi_\alpha$, determining the canonical variables for this problem have from (3:3) the solutions (3:2) with $\gamma_\alpha = 0$. Hence the Hamiltonian function for this problem is

$$H_1(x, y, z) = -F(x, y, y', \lambda) + y_1' F_{y_1},$$

in which $y_1' = H_{z_1}$, $\lambda_\alpha = \psi_\alpha$. From equations (3:6) we see that

$$H_1(x, y, z) = H[x, y, Z] - y_1' Z_1 + y_1' F_{y_1}, = H(x, y, z),$$

where in the second expression y_1' and λ_α are to be replaced as above, and the final equation is true because of equation (3:8).

We may now sum up these results in the following theorem:

THEOREM 3:1. If a function $H(x, y, z)$ is of class $C^{(4)}$ and has its matrix $\|H_{z_1 z_k}\|$ identically of rank $n - m$ in a region R of (xyz) space and if an arc E of class C' in R satisfies the differential equations (3:1), then it is possible to find an integrand function $f(x, y, y')$ and a set of differential equations $\phi_\alpha(x, y, y') = 0$ ($\alpha = 1, \dots, m$) such that a corresponding problem of Lagrange in the calculus of variations has the original function H as its Hamiltonian function in a neighborhood of the arc E .

Since the function H was assumed to be of class $C^{(4)}$, the second members of equations (3:2) are of class $C^{(3)}$ and hence also the second members of equations (3:3). Then by equations (3:6) and (3:7) the functions $F(x, y, y', \lambda)$ and $f(x, y, y')$ are of class $C^{(3)}$, a property which is customarily assumed for a problem in the calculus of variations. We note also that the arc E need not be an extremal. In fact it would be an extremal only if it satisfied the additional differential equations

$$dz_1/dx = -H_{z_1}(x, y, z).$$

Furthermore it is customarily assumed in a problem of Lagrange that the differential equations are independent, that is, that the matrix $\|\phi_{\alpha y_1}\|$ has rank m in the region where the problem

is defined. This property holds for the problem of Lagrange determined in the preceding paragraphs. To prove this we notice that if the functions (3:2) are substituted in equations (3:3), and the resulting identities differentiated with respect to the variables y_β , the equations

$$\phi_{\alpha y_k}, \psi_{\beta z_k} = \delta_{\alpha\beta}$$

are obtained. Suppose there exist solutions u_α of the equations $u_\alpha \phi_{\alpha y_k} = 0$. Then, multiplying the preceding equations by u_α and summing for α shows that $u_\alpha = 0$, and hence the matrix

$\|\phi_{\alpha y_1}\|$ must have rank m .

4. Equivalence of problems with the same Hamiltonian function. In the preceding section a method was given for finding a problem of Lagrange in the calculus of variations which has a given function $H(x, y, z)$ for its Hamiltonian function. A brief consideration of that method shows immediately that the problem of Lagrange there determined is not necessarily unique on account of the arbitrariness remaining in the choice of the polynomials $w_{1\beta}(x)$ in the argument preceding equations (3:2). It is natural then to investigate the relationship between different problems of Lagrange which have the same Hamiltonian function.

Let us recall that a problem of Lagrange in the calculus of variations is defined by a function $f(x, y, y')$ and a set of differential equations $\phi_\alpha(x, y, y') = 0$, where f and ϕ_α are of class $C^{(3)}$ in a suitable region S of the space of points (x, y, y') . Such a problem may be denoted by $P(f, \phi_\alpha)$. We now set up the following definition:

DEFINITION. Two problems of Lagrange $P(f, \phi_\alpha)$ and $P(g, \psi_\alpha)$ defined in the same region S of the space of points

(x, y, y') are equivalent provided that

(a) every admissible arc in S satisfying $\phi_\alpha = 0$, satisfies $\psi_\alpha = 0$ and conversely; and

(b) provided further that

$$\int_{x_1}^{x_2} f(x, y, y') dx = \int_{x_1}^{x_2} g(x, y, y') dx$$

along every admissible arc in S satisfying the equations $\phi_\alpha = 0$.

It is clear that if two problems are equivalent by this definition they determine the same minimizing and maximizing arcs. We now proceed to establish two lemmas.

LEMMA 4:1. Let $\phi_\alpha(x, y, y') = 0$ be a set of differential equations, the functions ϕ_α being of class $C^{(3)}$ and the matrix $\|\phi_{\alpha y_j}\|$ having rank m in a region S of points (x, y, y') . If $\psi(x, y, y') = 0$ is a differential equation, with ψ of class $C^{(3)}$ in S , such that every arc satisfying $\phi_\alpha = 0$, satisfies $\psi = 0$, then for each arc E of class C^1 in S satisfying the equations $\phi_\alpha = 0$ there exists a neighborhood S_1 and a set of functions $T_\alpha(x, y, y')$ of class $C^{(2)}$ in S_1 such that $\psi = T_\alpha \phi_\alpha$ in S_1 .

Let E be defined by functions $y_1(x)$ ($x_1 \leq x \leq x_2$). Then it is possible to select functions $\phi_\gamma(x, y, y')$ ($\gamma = m+1, \dots, n$) such that along E the determinant $|\phi_{1y_j}| \neq 0$ [4, pp. 307, 312]. Consider the equations

$$(4:1) \quad \phi_1(x, y, y') = z_1 \quad (i = 1, \dots, n)$$

which along E determine continuous functions $z_\alpha(x) = 0, z_\gamma(x)$.

By well known theorems concerning implicit functions we see readily that the equations (4:1) have solutions of the form

$$(4:2) \quad y_1' = w_1(x, y, z)$$

in which the functions $w_i(x, y, z)$ are of class $C^{(3)}$ in a neighborhood of the values $[x, y(x), z(x)]$ belonging to the arc E .

For $y_1 = y_1(x)$, $z_\alpha = 0$, $z_y = z_y(x)$, where $y_1(x)$, $z_1(x)$ are the functions belonging to E , and for $z_\alpha = 0$, they define elements (x, y, y') all of which satisfy $\phi_\alpha = 0$. Substitute now for y_1' from equations (4:2) in the function $\Psi(x, y, y')$ thus defining a function

$$\Psi(x, y, z) = \Psi[x, y, w(x, y, z)].$$

Obviously for $z_\alpha = 0$, we have $\Psi = 0$ since the equation $\Psi = 0$ is satisfied by all solutions of the equations $\phi_\alpha = 0$.

There exist functions $B_\alpha(x, y, z)$ defined and of class $C^{(2)}$ in a neighborhood of the values (x, y, z) belonging to the arc E and such that

$$(4:3) \quad \Psi = B_\alpha z_\alpha.$$

One such set is determined by applying Taylor's theorem with remainder in integral form to secure the equation

$$\Psi(x, y, z) = \Psi(x, y, z)|_{z_\alpha=0} + z_\alpha \int_0^1 \Psi_{z_\alpha}(x, y, tz_\alpha, z_y) dt.$$

Since the first term on the right vanishes identically it is clear that a set of functions with the required property is

$$B_\alpha = \int_0^1 \Psi_{z_\alpha}(x, y, tz_\alpha, z_y) dt.$$

If now we substitute in equations (4:3) the values of z_1 from equations (4:1) we find

$$\Psi(x, y, y') = T_\alpha(x, y, y') \phi_\alpha(x, y, y')$$

where

$$T_\alpha(x, y, y') = B_\alpha[x, y, \phi(x, y, y')]$$

and the functions T_α are of class $C^{(2)}$ in a suitably selected neighborhood S_1 of the values (x, y, y') on the arc E , as one readily verifies from the preceding argument.

LEMMA 4:2. Let $\phi_\alpha(x, y, y') = 0$, $\psi_\alpha(x, y, y') = 0$ ($\alpha = 1, \dots, m < n$) be two sets of differential equations such that the functions ϕ_α, ψ_α are of class $C^{(3)}$ and the matrices $\|\phi_{\alpha y_1}\|$, $\|\psi_{\alpha y_1}\|$ have rank m in a region S of points (x, y, y') . If every arc of class C' in S satisfying $\phi_\alpha = 0$ satisfies $\psi_\alpha = 0$ and conversely, then for each arc E of class C' in S satisfying $\phi_\alpha = 0$ there exists a neighborhood S_1 and a set of functions $T_{\alpha\beta}(x, y, y')$ of class $C^{(2)}$ in S_1 such that $\psi_\alpha = T_{\alpha\beta}\phi_\beta$, and the determinant $|T_{\alpha\beta}|$ is different from zero in S_1 wherever $\phi_\beta = 0$.

The existence of functions $T_{\alpha\beta}$ having the property

$\psi_\alpha = T_{\alpha\beta}\phi_\beta$ is an immediate consequence of Lemma 4:1. Furthermore we have

$$\psi_{\alpha y_1'} = T_{\alpha\beta}\phi_{\beta y_1'} + T_{\alpha\beta y_1'}\phi_\beta$$

and if $\phi_\alpha = 0$, $\psi_{\alpha y_1'} = T_{\alpha\beta}\phi_{\beta y_1'}$. For each element (x, y, y') satisfying $\phi_\beta = 0$, we may select a subset $y_{\beta'}$ of the variables y_1' such that the determinant $|\psi_{\alpha y_{\beta'}}| \neq 0$. Then $\psi_{\alpha y_{\beta'}} = T_{\alpha\gamma}\phi_{\gamma y_{\beta'}}$, and it follows at once that $|T_{\alpha\beta}| \neq 0$.

We are now in a position to prove the following theorem:

THEOREM 4:1. Two problems $P(f, \phi_\alpha)$, $P(g, \psi_\alpha)$ with the properties described above in a region S of points (x, y, y') are equivalent in S if and only if for each arc E of class C' in S satisfying the equations $\phi_\alpha = 0$ there is a neighborhood S_1 in which:

(a) there exist functions $T_{\alpha\beta}(x, y, y')$ of class $C^{(2)}$ in S_1 such that $\psi_\alpha = T_{\alpha\beta}\phi_\beta$ and $|T_{\alpha\beta}| \neq 0$ wherever $\phi_\alpha = 0$;

(b) there exist functions $S_\alpha(x, y, y')$ of class $C^{(2)}$ in
 S_1 such that

$$g(x, y, y') = f(x, y, y') + S_\alpha(x, y, y') \phi_\alpha(x, y, y').$$

The necessity of the existence of functions $T_{\alpha\beta}$ satisfying (a) is a consequence of Lemma 4:2. To prove (b) consider the function $h(x, y, y') = g(x, y, y') - f(x, y, y')$ which has the property that

$$\int_{x_1}^{x_2} h(x, y, y') dx = 0$$

along every arc of class C' in S_1 satisfying the equations

$\phi_\alpha = 0$, because of the definition of the equivalence of two problems. The preceding equation is then an identity in the upper limit of the integral when the arc in S_1 is assigned. Since a function which vanishes identically has an identically vanishing derivative, it follows that $h(x, y, y')$ must vanish along every arc in S_1 satisfying $\phi_\alpha = 0$. The condition (b) then follows from Lemma 4:1. That the conditions of the theorem are sufficient for the equivalence of the two problems is obvious.

We now prove one of the two principal results of this section.

THEOREM 4:2. Let $P(f, \phi_\alpha)$, $P(g, \psi_\alpha)$ be two problems of Lagrange defined in the same region S , and equivalent in S in the sense that they have functions $S_\alpha(x, y, y')$ and a non-singular matrix of functions $T_{\alpha\beta}(x, y, y')$ of class C'' in S and such that

$$\psi_\alpha = T_{\alpha\beta} \phi_\beta, \quad g = f + S_\alpha \phi_\alpha,$$

in S , and having Hamiltonian functions well defined in the regions T and U of points (x, y, z) which they make correspond to S .

Then the regions T and U are identical and the Hamiltonian functions of the two problems are identical in T.

When we say that the Hamiltonian function $H(x, y, z)$ of a problem $P(f, \phi_\alpha)$ is well defined in a region T of points (x, y, z) corresponding to S we mean that there is a region S_1 of points (x, y, y', λ) with S as its projection for $\lambda_\alpha = 0$, and such that the equations

$$(4:4) \quad z_1 = F_{y_1'} = f_{y_1'} + \lambda_\alpha \phi_{\alpha y_1'}, \quad v_\alpha = \phi_\alpha,$$

have a functional determinant

$$\begin{vmatrix} F_{y_1' y_k'} & \phi_{\beta y_1'} \\ \phi_{\alpha y_k'} & 0 \end{vmatrix}$$

with respect to the variables y_1', λ_α , different from zero in S_1 , and define a one-to-one correspondence between S_1 and the set T_1 of points (x, y, z, v) which they make correspond to S_1 . These equations then have solutions

$$(4:5) \quad y_1' = P_1(x, y, z, v), \quad \lambda_\alpha = L_\alpha(x, y, z, v)$$

which are of class $C^{(2)}$ in the region T_1 and in particular in the projection T of T_1 in the space $(x, y, z, v) = (x, y, z, 0)$. In this case the Hamiltonian function for the problem $P(f, \phi_\alpha)$ is defined by means of the equation

$$H(x, y, z) = -F(x, y, y', \lambda) + y_1' F_{y_1'},$$

with the variables y_1', λ_α replaced on the right by means of equations (4:5) with $v_\alpha = 0$.

For the problem $P(g, \psi_\alpha)$, we have the following relation,

$$\begin{aligned} G(x, y, y', \mu) &= g + \mu_\beta \psi_\beta = f + S_\alpha \phi_\alpha + \mu_\beta T_{\alpha\beta} \phi_\beta \\ &= f + \lambda_\alpha \phi_\alpha = F(x, y, y', \lambda) \end{aligned}$$

where

$$(4:6) \quad \lambda_\alpha = S_\alpha + \mu_\beta T_{\alpha\beta}.$$

Also the equations corresponding to (4:4) are

$$(4:7) \quad z_1 = G_{y_1'} = F_{y_1'} + F_{\lambda_\alpha} \lambda_{\alpha y_1'}, \quad \nu_\alpha = \psi_\alpha = T_{\alpha\beta} \phi_\beta.$$

Consider an element (x, y, y', λ) in the region S_1 for the problem $P(f, \phi_\alpha)$ satisfying the equations $\phi_\alpha = 0$, and the element (x, y, y', μ) in the analogous region $\bar{S}_1$ for the problem $P(g, \psi_\alpha)$ corresponding to (x, y, y', λ) by means of equations (4:6). The corresponding points (x, y, z) for the two problems are both defined by equations (4:7). Since every element (x, y, z) in T is the image of such an element (x, y, y', λ) it follows that every element (x, y, z) of T is an element of U , and a similar argument shows that the converse is true, so that T and U are identical. Furthermore when the variables λ_α and μ_α are related by the equations (4:6) the equations (4:4) and (4:7) with $\nu_\alpha = 0$ are identical. We see also that the solutions (4:5) of the equations (4:4) give the corresponding solutions for y_1' , μ_α , of equations (4:7), where again we relate the variables λ_α and μ_α by means of equations (4:6). It follows readily that the Hamiltonian functions $H(x, y, z)$ for the two problems are identical. It should be noticed that while the hypotheses of the theorem ensure equivalence of the two problems in S according to the definition, yet it cannot be shown by the methods here used that these hypotheses are a consequence of the definition.

THEOREM 4:3. Let $P(f, \phi_\alpha)$ and $P(g, \psi_\alpha)$ be two problems of Lagrange defined in the same region S of points (x, y, y') as

described above, and having the same regions T and Hamiltonian functions $H(x, y, z)$ defined in T. Then the two problems are equivalent in the sense defined above.

For the two problems under consideration we have the equations

$$(4:8) \quad z_1 = F_{y_1'}(x, y, y', \lambda), \quad \mathcal{V}_\alpha = \Phi_\alpha(x, y, y')$$

$$(4:9) \quad z_1 = G_{y_1'}(x, y, y', \mu), \quad \mathcal{V}_\alpha = \Psi_\alpha(x, y, y')$$

and their solutions for the variables y_1' , λ_α and y_1' , μ_α ,

$$(4:10) \quad \begin{aligned} y_1' &= P_1(x, y, z, \nu), & \lambda_\alpha &= L_\alpha(x, y, z, \nu), \\ y_1' &= Q_1(x, y, z, \nu), & \mu_\alpha &= M_\alpha(x, y, z, \nu). \end{aligned}$$

We introduce functions $H_1(x, y, z, \nu)$, $H_2(x, y, z, \nu)$ by the equations

$$(4:11) \quad \begin{aligned} H_1(x, y, z, \nu) &= -F(x, y, P, L) + z_1 P_1 + \mathcal{V}_\alpha L_\alpha \\ H_2(x, y, z, \nu) &= -G(x, y, Q, M) + z_1 Q_1 + \mathcal{V}_\alpha M_\alpha \end{aligned}$$

which, by hypothesis, satisfy the relation

$$H_1(x, y, z, 0) = H_2(x, y, z, 0) = H(x, y, z)$$

identically for (x, y, z) in the region T. A simple differentiation shows that $H_{1z_1} = P_1$, $H_{2z_1} = Q_1$, so that the equations (4:10) are equivalent to the equations

$$(4:12) \quad y_1' = H_{1z_1}(x, y, z, \nu), \quad \lambda_\alpha = L_\alpha(x, y, z, \nu),$$

$$(4:13) \quad y_1' = H_{2z_1}(x, y, z, \nu), \quad \mu_\alpha = M_\alpha(x, y, z, \nu).$$

Let us now consider the equations (4:8), (4:9), (4:12) and (4:13). Equations (4:12) and (4:13) give solutions for y_1' , λ_α of equations (4:8) and (4:9) respectively, and the right mem-

bers are defined for (x, y, z, v) in the region T_1 . Similarly equations (4:8) and (4:9) are solutions for z_1, v_α of equations (4:12) and (4:13) respectively, defined for (x, y, y', λ) in the region S_1 , and (x, y, y', μ) in $\bar{S}_1$. If an element (x, y, y', λ) satisfies $\phi_\alpha = 0$, it determines by equations (4:8) an element (x, y, z, v) with $v_\alpha = 0$. These two elements are also related by equations (4:12) with $v_\alpha = 0$, namely

$$y_1' = H_{1z_1}(x, y, z, 0) = H_{z_1}(x, y, z).$$

By means of equations (4:13) the element $(x, y, z, 0)$ determines an element (x, y, y', μ) where the (x, y, y') are the same as for the original element, but the λ_α and μ_α are in general distinct. These last two elements are also connected by equations (4:9) with $v_\alpha = 0$ and hence we see that the original values (x, y, y') satisfy $\psi_\alpha = 0$ as well as $\phi_\alpha = 0$. Therefore every set of values (x, y, y') satisfying $\phi_\alpha = 0$ satisfies $\psi_\alpha = 0$ and conversely.

Now let us consider an element $(x, y, z, v) = (x, y, z, 0)$ and the corresponding elements (x, y, y', λ) , (x, y, y', μ) determined by means of equations (4:12) and (4:13). Since $v_\alpha = 0$, the values (x, y, y') satisfy the equations $\phi_\alpha = 0$, $\psi_\alpha = 0$. Furthermore from the preceding paragraphs it follows that

$$P_1(x, y, z, 0) = H_{1z_1}(x, y, z, 0) = H_{z_1}(x, y, z),$$

$$Q_1(x, y, z, 0) = H_{2z_1}(x, y, z, 0) = H_{z_1}(x, y, z).$$

These equations taken together with equations (4:11) and the hypothesis that the two Hamiltonian functions are identical imply that $f(x, y, H_z) = g(x, y, H_z)$ identically for (x, y, z) in the region T . But for every element (x, y, y') satisfying the equations $\phi_\alpha = 0$, the equations (4:8), or (4:9) define a set z_1 such that $y_1' = H_{z_1}(x, y, z)$. It follows readily that the equation

$f(x, y, y') = g(x, y, y')$ must hold for every set (x, y, y') satisfying the equations $\Phi_\alpha = 0$. The two conditions in the definition of equivalence are thus satisfied, and the proof is completed.

5. An application to the inverse problem of the calculus of variations. The inverse problem of the calculus of variations in three dimensional space is well known to be that of finding the problems in the calculus of variations for which a given four parameter family of curves is the family of extremals. The preceding results show that a problem of the calculus of variations having such a family of extremals would be determined if we could find a Hamiltonian function $H(x, y, z, u, v)$ such that the given curves

$$(5:1) \quad y = y(x, a, b, c, d), \quad z = z(x, a, b, c, d)$$

would be with two other functions

$$u = u(x, a, b, c, d), \quad v = v(x, a, b, c, d)$$

the complete family of solutions of the canonical equations

$$y' = H_u, \quad z' = H_v, \quad u' = -H_y, \quad v' = -H_z.$$

A Hamiltonian function $H(x, y, z, u, v)$ in fact determines a problem in the calculus of variations with the solutions of the canonical equations as extremals, as was shown in section 3 above, and the problem will have no or one differential equation as a side condition according as the matrix

$$\begin{vmatrix} H_{uu} & H_{uv} \\ H_{uv} & H_{vv} \end{vmatrix}$$

is of rank two or one.

In attempting to solve this inverse problem, we are soon faced with the same difficulties that Davis [6] in his dissertation of 1927 was unable to overcome completely. However, we may obtain some further interesting partial results by a consideration of the Hamilton-Jacobi partial differential equation.

It is well known [2, p. 601] that if the function $W(x, y, z, a, b) + k$ is a complete integral of the Hamilton-Jacobi partial differential equation

$$(5:2) \quad W_x + H(x, y, z, W_y, W_z) = 0$$

for a problem in the calculus of variations with Hamiltonian function $H(x, y, z, u, v)$, then the extremals for the problem are given by the equations

$$(5:3) \quad W_a = c, \quad W_b = d,$$

where c and d are arbitrary constants like a and b . Conversely, suppose we are given a function $W(x, y, z, a, b)$ of class $C^{(2)}$ and having a determinant

$$\begin{vmatrix} W_{ay} & W_{az} \\ W_{by} & W_{bz} \end{vmatrix}$$

different from zero in a neighborhood of a region R of points (x, y, z, a, b) . If we set

$$(5:4) \quad W_y(x, y, z, a, b) = u, \quad W_z(x, y, z, a, b) = v,$$

then according to well known implicit function theorems these equations have solutions for the variables a and b , of class $C^{(2)}$ in a neighborhood of the set S of points (x, y, z, u, v) corresponding to the set R by means of equations (5:3). If we denote

these solutions by

$$(5:5) \quad a = A(x, y, z, u, v), \quad b = B(x, y, z, u, v)$$

we may then define a function H by means of the equation

$$(5:6) \quad H(x, y, z, u, v) = -W_x[x, y, z, A(x, y, z, u, v), B(x, y, z, u, v)].$$

Obviously since the equations (5:4) and (5:5) are mutually solutions of each other, the given function $W(x, y, z, a, b)$ is a complete integral of the Hamilton-Jacobi partial differential equation (5:2) containing this function H , and hence the curves (5:3) are the extremals for every problem of the calculus of variations having the function H as its Hamiltonian function. We may now state the theorem:

THEOREM 5:1. In order that a four-parameter family of curves may be the extremals of a problem in the calculus of variations, it is necessary and sufficient that there exist a function $w(x, y, z, a, b)$ with continuity properties as described above and such that the given curves are expressible in the form (5:3). Under these conditions the curves are the extremals of every problem of the calculus of variations related as in Section 3 to the Hamiltonian function H which is determined as described above by means of equations (5:4), (5:5), (5:6).

In order to give a complete treatment of the inverse problem from this approach, it would be necessary to give explicit conditions on the functions (5:1) in order that they might be expressible in the form (5:3). This however does not appear to be an easy task.

It will be of interest to approach the problem from a slightly different point of view. A four parameter family of curves may be defined by means of two slope functions each con-

taining two arbitrary parameters. If the slope functions are $P(x, y, z, a, b)$, $Q(x, y, z, a, b)$ the four parameter family of curves will be solutions of the differential equations

$$(5:7) \quad y' = P(x, y, z, a, b), \quad z' = Q(x, y, z, a, b).$$

For fixed values of a and b , the functions P and Q would not in general be the slope functions of a field of extremals. We may now prove the following theorem:

THEOREM 5:2. If a function $W(x, y, z, a, b)$ of class $C^{(2)}$ and having the determinant $W_{ay}W_{bz} - W_{az}W_{by}$ different from zero in a region R of points (x, y, z, a, b) , satisfies equation (5:2) with the Hamiltonian function H of a problem in the calculus of variations, then there exist slope functions $P(x, y, z, a, b)$, $Q(x, y, z, a, b)$, of class C^1 in a region S contained in R , and satisfying the equations

$$(5:8) \quad W_{ax} + W_{ay}P + W_{az}Q = 0, \quad W_{bx} + W_{by}P + W_{bz}Q = 0$$

identically for (x, y, z, a, b) in S . Conversely if there exist functions P , Q and W with the above continuity properties and satisfying equations (5:8), the equations (5:7) determine a four parameter family of extremals of a problem in the calculus of variations.

To prove the existence of such slope functions for a problem in the calculus of variations let us define functions P and Q by means of the equations

$$P(x, y, z, a, b) = H_u(x, y, z, W_y, W_z),$$

$$Q(x, y, z, a, b) = H_v(x, y, z, W_y, W_z).$$

Since the function H is of class $C^{(2)}$ in a region T of points

(x, y, z, u, v) , the functions P and Q will be of class C^1 in the region S of points (x, y, z, a, b) for which the sets (x, y, z, W_y, W_z) are in T . The equations (5:8) are an immediate consequence of differentiating with respect to a and b the equation (5:2) considered as an identity in the variables (x, y, z, a, b) .

For the converse we notice that equations (5:8) imply that along the solutions of equations (5:7), W_a and W_b are constant. Hence, Theorem (5:1) immediately insures the desired conclusion.

If now we were given a four parameter family of curves defined by differential equations (5:7), and if we desired to find the conditions on the functions P and Q in order that the given curves might be the extremals of a problem in the calculus of variations, it would be sufficient to find the conditions which would guarantee the existence of a solution $W(x, y, z, a, b)$ of the equations (5:8). Instead of considering the pair of second order partial differential equations, we might examine the system

$$(5:9) \quad M_x + M_y P + M_z Q = 0, \quad N_x + N_y P + N_z Q = 0, \quad M_b = N_a.$$

In either case, it appears to the present writer that more knowledge concerning systems of partial differential equations than is at present available is required before a complete solution of the problem will be obtained. At the present time it appears more fruitful to examine another feature of problems in the calculus of variations, namely the transversality. This has already been done by Douglas and Rawles, and we shall investigate that problem further in the following sections.

6. Duality between problems of the calculus of variations and infinitesimal contact transformations. In this section we show that it is always possible to associate with a given problem in the calculus of variations an infinitesimal contact transformation whose transversality and path curves are identical with the transversality and extremals of the problem in the calculus of variations. The treatment here is more complete and more direct than that employed by Douglas [7, pp. 401-409]. Furthermore all such infinitesimal contact transformations are found.

A transversality T in $n + 1$ dimensional space may be defined as a set of relations

$$(6:1) \quad p_1 = p_1(x, y, y')$$

connecting line elements (x, y, y') and hypersurface elements (x, y, p) having the same base point (x, y) . For all transversalities treated in the following paragraphs it will be assumed that the equations (6:1) have solutions

$$(6:2) \quad y_1' = y_1'(x, y, p)$$

for the variables y_1' , and further that there is a region R in the space of $2n + 1$ dimensional points (x, y, y') in which

$$(6:3) \quad 1 - y_1' p_1(x, y, y') \neq 0.$$

Let us consider the problem in the calculus of variations

$$(6:4) \quad I = \int_{x_1}^{x_2} f(x, y, y') dx = \text{minimum},$$

to which we shall refer as the problem f . A hypersurface $x = x(y_1)$ is said to be transversal [9, p. 105] to the direction $(1, y_1')$ in case

$$dx(y_1' f_{y_1} - f) - f_{y_1} dy_1 = 0$$

for every set of differentials (dx, dy_1) satisfying the relation

$$dx - \frac{\partial x}{\partial y_1} dy_1 = dx - p_1 dy_1 = 0.$$

This definition sets up the transversality

$$(6:5) \quad p_1 = f_{y_1} / (y_1' f_{y_1} - f)$$

which will be referred to as the transversality belonging to the problem f . Douglas bases his treatment of this subject on the differential form $dx + p_1 dy_1 = 0$ [7, p. 402] thus having to take his surface in the form $x = -x(y_1)$. The method chosen here seems to be the natural one. Obviously it will be necessary to assume that the expression $(y_k' f_{y_k} - f)$ is always different from zero. Furthermore for the transversality defined by means of equations (6:5), we have

$$1 - p_k y_k' = -f / (y_k' f_{y_k} - f)$$

which satisfies the condition (6:3), providing that $f \neq 0$.

A general transversality (6:1) will belong to a problem f in the calculus of variations if and only if f is a non-vanishing solution of the linear partial differential equation of the first order

$$(6:6) \quad p_1(x, y, y')f + (\delta_{1k} - p_1 y_k') f_{y_k} = 0,$$

this being an immediate consequence of the equations (6:1), (6:5). These equations can be converted into homogeneous equations by introducing a function $F(x, y, y', f)$ such that $f(x, y, y')$ is a solution of the equations (6:6) if and only if it is a solution of an equation of the form $F[x, y, y', f(x, y, y')] = 0$ identically

in (x, y, y') . The corresponding homogeneous equations are then

$$(6:7) \quad U_1 F = p_1 f F_f - (\delta_{1k} - p_1 y_k') F_{y_k'} = 0.$$

The theory of simultaneous homogeneous linear partial differential equations of the first order assures us that there exist non-constant solutions F if and only if the system (6:7) is complete, that is, if and only if its commutators are expressible as linear combinations of the expressions $U_1 F$ [3, pp. 265-271]. The commutators are easily calculated to be

$$\begin{aligned} (U_1 U_j) F = & F_f [p_1 y_{j'} - p_{j y_1'} + y_h' (p_1 p_{j y_h'} - p_j p_{1 y_h'})] \\ & + F_{y_k'} [(p_1 \delta_{kj} - p_j \delta_{k1}) + y_k' (p_1 y_{j'} - p_{j y_1'}) \\ & + y_k' y_h' (p_1 p_{j y_h'} - p_j p_{1 y_h'})]. \end{aligned}$$

These will be linear combinations of the expressions $U_1 F$ if and only if the $n + 1$ rowed determinant

$$D = \begin{vmatrix} A & p_m \\ B & p_m y_k' - \delta_{mk} \end{vmatrix},$$

where

$$A = (p_1 y_{j'} - p_{j y_1'}) + y_h' (p_1 p_{j y_h'} - p_j p_{1 y_h'}),$$

$$B = (p_1 \delta_{kj} - p_j \delta_{k1}) + y_k' (p_1 y_{j'} - p_{j y_1'}) + y_k' y_h' (p_1 p_{j y_h'} - p_j p_{1 y_h'}),$$

vanishes for each fixed i and j . To reduce D subtract from the $(k + 1)$ -st row y_k' times the first row. Then multiply the $(k + 1)$ -st row by p_k , sum for k and add to the first row. This shows that

$$D = \begin{vmatrix} A & 0 \\ p_1 \delta_{kj} - p_j \delta_{k1} & \delta_{mk} \end{vmatrix}$$

and hence $D = 0$ if and only if the functions $p_1(x, y, y')$ satisfy

the conditions

$$(6:8) \quad M_{1j}(x, y, y'; p, p_{y'}) = p_{1y_j'} - p_{jy_1'} + y_k'(p_1 p_{jy_k'} - p_j p_{1y_k'}) = 0.$$

The notation is chosen to indicate that the independent variables are (x, y, y') while the expressions M_{1j} depend also on the functions p_1 and their derivatives $p_{1y_j'}$. Suppose now that the conditions (6:8) are satisfied and that $F(x, y, y', f)$ is a particular integral of the equations (6:7). The theory tells us that every solution is an arbitrary function of F . To obtain all solutions of equation (6:6) we set $\phi(F) = 0$, where ϕ is an arbitrary function or, what amounts to the same thing, we set $F = g(x, y)$ and solve for f .

Instead of using the general theory, we might have proceeded directly. First, we notice that the determinant

$$|\delta_{ik} - a_1 b_k| \text{ has the value}$$

$$|\delta_{ik} - a_1 b_k| = 1 - a_1 b_1.$$

This statement may be verified by a simple induction. The equations (6:6) may be considered as linear equations in the ratios $f_{y_1'}/f$. The determinant of coefficients is

$$|\delta_{ik} - p_1 y_k'| = 1 - p_k y_k'$$

which is different from zero. Hence there exist solutions for $f_{y_1'}/f$, and by direct substitution it may be verified that these solutions are

$$f_{y_1'}/f = -p_1/(1 - y_k' p_k).$$

These equations have non-vanishing solutions f if and only if

$$\frac{\partial}{\partial y_j'} \frac{-p_1}{1 - p_k y_k'} = \frac{\partial}{\partial y_1'} \frac{-p_j}{1 - p_k y_k'},$$

and upon differentiation these conditions become

$$(6:10) \quad R_{ij}(x, y, y'; p, p_y) = \\ p_i y_j' - p_j y_i' + y_k'(p_i p_{ky_1}' - p_j p_{ky_1}' + p_k p_{jy_1}' - p_k p_{iy_j}') = 0.$$

When the conditions (6:10) are satisfied

$$(6:11) \quad \log f = \int \frac{-p_i dy_1'}{1 - p_k y_k'} + \log g(x, y), \\ f = g e^{\theta}, \quad \theta = \int \frac{-p_i dy_1'}{1 - p_k y_k'}$$

where g is an arbitrary function of (x, y_1) and (x, y_1) are to be considered constant during the integration. This is precisely the result obtained by Rawles [8, pp. 774-778] but here it is obtained more directly.

Obviously conditions (6:8) and (6:10) must be equivalent. In fact a simple calculation shows that

$$R_{ij} = M_{ij}(1 - p_k y_k') + p_i y_k' M_{kj} + p_j y_k' M_{ik} \\ (1 - p_k y_k') M_{ij} = R_{ij} - p_i y_k' R_{kj} - p_j y_k' R_{ik}$$

so that $R_{ij} = 0$ if and only if $M_{ij} = 0$. We may now sum up these results in the following theorem:

THEOREM 6:1. A transversality defined by means of equations (6:1) and satisfying the conditions (6:3) belongs to a problem in the calculus of variations if and only if the conditions (6:8) or (6:10) are satisfied by the functions p_i and their derivatives p_{iy_j}' . When these conditions hold, the transversality belongs to the problem f of the calculus of variations, with f defined by means of the equations (6:11), or by means of an equation $F(x, y, y', f) = g(x, y)$, where F is a particular integral

of the system (6:7) and g is an arbitrary function of the variables (x, y_1) .

The general infinitesimal contact transformation in $n + 1$ dimensional space may be represented analytically in explicit form by means of its characteristic function $W(x, y, p)$ as follows [1, p. 252]:

$$dx = dt(p_1 W_{p_1} - W), \quad dy_1 = W_{p_1} dt, \quad dp_1 = -dt(p_1 W_x + W_{y_1}).$$

For such an infinitesimal contact transformation a transversality is defined by means of the equations

$$(6:12) \quad y_1' = W_{p_1} / (p_1 W_{p_1} - W),$$

where it must be assumed that $p_1 W_{p_1} - W$ is always different from zero. We shall speak of a transversality having the form (6:12) as belonging to an infinitesimal contact transformation W . It is obvious by a simple change of notation, on account of the analogy between (6:1), (6:2) and (6:5), (6:12) that the following theorem, analogous to Theorem (6:1), holds:

THEOREM 6:2. A transversality defined by means of equations (6:2) and satisfying condition (6:3) thought of as a condition in the variables (x, y, p) , belongs to an infinitesimal contact transformation W if and only if $M_{1j}(x, y, p; y', y_p') = 0$ or $R_{1j}(x, y, p; y', y_p') = 0$. When these conditions hold the transversality belongs to the infinitesimal contact transformation W where W is defined by the equations

$$(6:13) \quad W = h e^{\phi}, \quad \phi = \int -y_1' dp_1 / (1 - p_k y_k'),$$

h being an arbitrary function of (x, y_1) , and these variables being considered constant in the integration.

Suppose now that we are given a general transversality defined by the equation (6:1) or their solutions (6:2) for the variables y_1' . Then we have

$$p_1 = p_1[x, y, y'(x, y, p)], \quad y_1' = y_1'[x, y, p(x, y, y')]$$

identically in (x, y, p) , (x, y, y') respectively, and by differentiation

$$\delta_{ij} = p_{1y_k} y_{kp_j}' = y_{1p_k} p_{ky_j}'.$$

By making use of these second identities, a direct calculation shows that the following relations hold:

$$\begin{aligned} p_{1y_k} M_{kh}(x, y, p; y', y_p') p_{jy_h}' \Big|_{p=p(x, y, y')} \\ = -M_{1j}(x, y, y'; p, p_y), \\ y_{1p_k} M_{kh}(x, y, y'; p, p_y) y_{jp_h}' \Big|_{y'=y'(x, y, p)} \\ = -M_{1j}(x, y, p; y', y_p'). \end{aligned}$$

Thus $M_{1j}(x, y, y'; p, p_y) = 0$ if and only if $M_{1j}(x, y, p; y', y_p') = 0$. In view of Theorems (6:1) and (6:2), the preceding remark justifies the first statement in the following theorem:

THEOREM 6:3. If a transversality belongs to a problem of the calculus of variations it belongs also to an infinitesimal contact transformation, and conversely. Furthermore, in order that a transversality relation belonging to a given problem of the calculus of variations with integrand function $f(x, y, y')$ shall belong also to an infinitesimal contact transformation with generating function $W(x, y, p)$, it is necessary and sufficient that f and W satisfy the relation

$$(6:14) \quad fW = K(x, y)(1 - p_k y_k')$$

which is to be thought of as an identity in (x, y, p) or (x, y, y') when y' or p , respectively, are replaced by means of equations defining the transversality.

To prove the necessity in the second statement consider a transversality (6:1) or (6:2) which satisfies the conditions (6:8). Every problem in the calculus of variations whose transversality is the given one has the form (6:11), while every infinitesimal contact transformation with this transversality has the form (6:13). We have then directly

$$fW = g(x, y)h(x, y)e^{\theta + \phi},$$

$$\theta + \phi = \int \frac{-p_1 dy_1' - y_1' dp_1}{1 - p_1 y_1'} = \log [(1 - p_k y_k')A(x, y)]$$

and the relation (6:14) is seen to hold with $K = ghA$.

Conversely, if (6:14) holds we wish to show that the equations (6:5) defining the transversality of the problem with integrand function f , have as their solutions for y_1' the expressions (6:12) which define the transversality of the infinitesimal contact transformation W . For convenience we will use the notation

$$H = y_1' f_{y_1'} - f, \quad M = p_1 W_{p_1} - W.$$

If we substitute from the equations (6:5) in equation (6:14) we get $W = -K/H$, and substituting this back into equations (6:5) we find

$$(6:15) \quad f_{y_1'} = -p_1 K/W.$$

Consider the relation (6:14) as an identity in the variables (x, y, y') and differentiate it with respect to y_1' . After applying equation (6:15) and using the fact that the determinant $|p_{1y_k}|$ is different from zero we get $y_1' = -fW_{p_1}/K$. Substitute

this expression for y_k' in equation (6:14), solve for f , and put this expression for f in the preceding expression for y_1' . This leads to the equation (6:12). Hence the theorem is completely justified.

Let us now seek necessary and sufficient conditions on the function $K(x, y)$ in order that the extremals of the problem f of the calculus of variations shall coincide with the path curves of the infinitesimal contact transformation W . To do this we shall consider the equations (6:5) as equations introducing canonical variables (x, y, p) and calculate the corresponding canonical equations for the extremals. Differentiation of equation (6:14) with respect to y_1 leads to the expression

$$f_{y_1} = (W_{y_1} K - K_{y_1} W) / WM.$$

This relation together with relation (6:15) now give the following:

$$\begin{aligned} 0 &= \frac{d}{dx}(f_{y_1}') - f_{y_1} = \frac{d}{dx}(-p_1 K / W) - (W_{y_1} K - K_{y_1} W) / WM \\ &= -p_j' \frac{K}{W} (\delta_{1j} - p_1 W_{p_j} / W) + \frac{p_1 K}{W} (W_x / W + W_{y_j} W_{p_j} / MW - K_x - K_{y_j} W_{p_j} / M) \\ &\quad - K W_{y_1} / MW + K_{y_1} / M. \end{aligned}$$

The determinant of coefficients of the variables p_j' can be evaluated by an earlier remark, its value being $-M/W$ which we have previously assumed to be different from zero. Hence there exist unique solutions for the variables p_1' . The differential equations for the path curves are

$$(6:16) \quad y_1' = W_{p_1} / M, \quad p_1' = -(p_1 W_x + W_{y_1}) / M.$$

In order that the previously mentioned solutions for p_1' shall be identical with the above expressions, it is necessary and suffi-

cient that the result of substituting for p_1' from equations (6:16) shall reduce to an identity. Actual substitution shows that the coefficient of K vanishes and the resulting identity in the variables (x, y, p) is

$$p_1(p_j^W p_j - W)K_x - (\delta_{1j} - p_1^W p_j / W)WK_{y_j} = 0.$$

In these equations the determinant of coefficients of the derivatives K_{y_j} is different from zero and there exist unique solutions for these derivatives. By actual substitution we can verify that these unique solutions are $K_{y_j} = -p_j K_x$. But we have assumed that K , and hence K_y and K_x also are independent of p . Thus we have reached a contradiction unless $K_x = K_{y_j} = 0$, from which it follows immediately that $K = \text{constant}$. Obviously if $K = \text{constant}$ the identity is satisfied. We have now proved the theorem

THEOREM 6:4. The transversality and extremals of a problem f in the calculus of variations coincide with the transversality and path curves of an infinitesimal contact transformation W if and only if

$$fW = c(1 - p_k y_k')$$

which is to be considered as an identity in the variables (x, y, p) or (x, y, y') where y' or p is replaced by means of the transversality relations.

There will be no loss of generality if we always take $c = 1$, since c can be absorbed into W without changing either the transversality or the path curves.

It is of interest to notice that there is an important parallel between the generating function W of the infinitesimal contact transformation, and the Hamiltonian function H . In fact, in order to define H for a problem of the calculus of variations

we define canonical variables z_1 by means of the equations

$$(6:17) \quad z_1 = f_{y_1'}(x, y, y')$$

and define H by means of the equation

$$f + H = y_1' z_1$$

where the variables y_1' are to be replaced by the solutions for y_1' of the equations (6:17). In terms of the canonical variables (x, y, z) the differential equations of the extremals are then the canonical equations

$$dy_1/dx = H_{z_1}, \quad dz_1/dx = -H_{y_1}.$$

In order that a $2n$ -parameter family of curves defined in (xyz) space by differential equations of the form

$$dy_1/dx = y_1'(x, y, z), \quad dz_1/dx = z_1'(x, y, z)$$

shall be the family of extremals in canonical variables of a problem of the calculus of variations it is necessary and sufficient that the conditions

$$\frac{\partial y_1'}{\partial z_k} = \frac{\partial y_k'}{\partial z_1}, \quad \frac{\partial y_1'}{\partial y_k} = \frac{\partial z_1'}{\partial z_k}, \quad \frac{\partial z_1'}{\partial y_k} = \frac{\partial z_k'}{\partial y_1}$$

shall be satisfied. However, if canonical variables are introduced by the equations (6:5) and if a function W is defined by means of the equation

$$fW = 1 - p_k y_k'$$

with the variables y_1' replaced by the solutions for y_1' of equations (6:5), the function W is then the generating function of an infinitesimal contact transformation. Furthermore, as we have shown above, the canonical equations for the extremals with this

set of canonical variables (x, y, p) are simply the equations (6:16) of the path curves of the infinitesimal contact transformation.

7. Analytic characterization of the composite of extremals and transversality. In this section we seek to determine necessary and sufficient conditions that a $2n$ parameter family of curves

$$(7:1) \quad y_1 = y_1(x, a, b) \quad (x_1 \leq x \leq x_2),$$

and a transversality

$$(7:2) \quad y_1' = y_1'(x, y, p)$$

shall be the extremals and transversality of a problem in the calculus of variations. If we solve the equations

$$(7:3) \quad y_{1x}(x, a, b) = y_1' [x, y(x, a, b), p]$$

for the variables p_1 , we obtain n functions

$$p_1 = p_1(x, a, b)$$

which, when adjoined to the functions (7:1) form a system of solutions of a set of differential equations

$$(7:4) \quad y_1' = y_1'(x, y, p), \quad p_1' = p_1'(x, y, p),$$

obtained by eliminating the parameters a, b . The functions $y_1'(x, y, p)$ are necessarily those on the right of equations (7:2) on account of the relations (7:3).

By the results of the preceding section it is evident that in order to determine whether or not the extremals (7:1) and the transversality (7:2) are the extremals and transversality of a problem of the calculus of variations it will be sufficient to determine whether or not the $2n$ parameter family of curves

$$y_1 = y_1(x, a, b), \quad p_1 = p_1(x, a, b)$$

in (x, y, p) space is the family of path curves of an infinitesimal contact transformation. The differential equations (7:3) will define the path curves of an infinitesimal contact transformation with generating function $W(x, y, p)$ if and only if there exist solutions W of the equations

$$(7:5) \quad \begin{aligned} y_1'(x, y, p) &= W_{p_1} / (p_j W_{p_j} - W), \\ p_1'(x, y, p) &= -(p_1 W_x + W_{y_1}) / (p_j W_{p_j} - W). \end{aligned}$$

Obviously the transversality (7:2) must satisfy the conditions developed in the preceding section from the first n of the equations (7:5). It is there shown that when these conditions are satisfied the first n of equations (7:5) have a solution of the form

$$(7:6) \quad W = h(x, y)e^{\phi}, \quad \phi(x, y, p) = \int -y_1' dp_1 / (1 - p_k y_k').$$

It now remains to determine the conditions on the functions $p_1'(x, y, p)$ in order that there may exist a function $h(x, y, p)$ depending upon the variables p as well as x and y which will make the function W defined in equations (7:6) a solution of the equations (7:5). If we substitute for W from equations (7:6) in equations (7:5) we see that these will be satisfied if and only if $h(x, y, p)$ is a solution of the system of linear partial differential equations

$$(7:7) \quad \begin{aligned} p_1 h_x + h_{y_1} + h[p_1 \phi_x + \phi_{y_1} - p_1'(x, y, p) / (1 - p_k y_k')] &= 0, \\ h_{p_1} &= 0. \end{aligned}$$

These equations can be satisfied by a function $h(x, y)$ if and only if the functions

$$(7:8) \quad A_1 = -p_1 \phi_x - \phi_{y_1} + p_1'(x, y, p)/(1 - p_k y_k')$$

have the form

$$(7:9) \quad A_1 = U_x p_1 + U_{y_1}$$

where U is a function of x, y only. Hence we have the following theorem:

THEOREM 7:1. A $2n$ parameter family of curves (7:1) and a transversality (7:2) will be the composite of extremals and transversality for an infinitesimal contact transformation, and hence too for a problem in the calculus of variations if and only if the transversality satisfies the conditions of Theorem 6:1 or 6:2, and the functions $p_1'(x, y, p)$ of equations (7:4), obtained by eliminating the parameters a and b from equations (7:1) and (7:2) are such that the expressions A_1 in equations (7:8) have the form (7:9). In this case the corresponding infinitesimal contact transformation has as its generating function $W = h e^\phi$ where

$$\phi = \int -y_1' dp_1 / (1 - p_k y_k')$$

and h is a non-vanishing solution of the system (7:7), while the corresponding problem in the calculus of variations is obtained as shown in Theorem 6:4.

The problem of setting down explicitly the analytic conditions that a $2n$ -parameter family of curves of the form (7:1) shall have associated with it a transversality such that the family and the transversality are the extremals and transversality of a problem in the calculus of variations seems to be still unsolved.

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